MATH 2204 **Sample Exam 1**

Instructor: Phanuel Mariano

Name: KEY

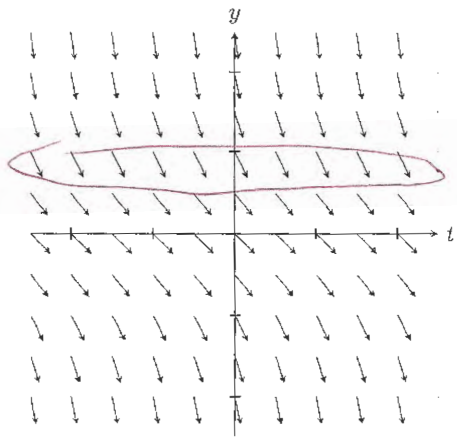
Instructions:

- All answers must be written clearly.
- You may use a calculator (TI-84 or below), but you must show all your work in order to receive credit. This includes any multiple choice questions! No credit will be given to any problem unless work is shown.
- Be sure to erase or cross out any work that you do not want graded.
- If two answers are circled in the multiple choice, then zero credit is given.
- If you need extra space, you may use the back sides of the exam pages (if you do, please write me a note so that I know where to look).
- Any cheating will result in an immediate F in the course.
- Partial credit will be given to open ended problems.
- **The ACTUAL EXAM will not be this long! This is just to give you an idea of the type and style of questions to be asked**

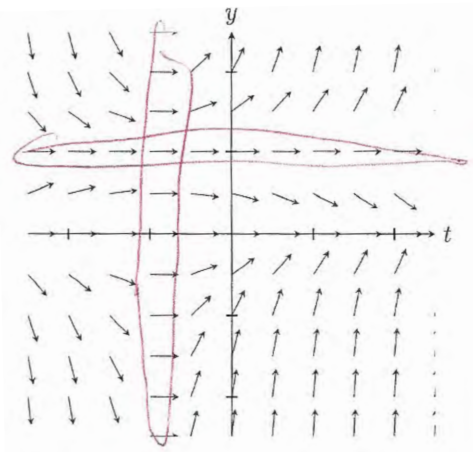
Question:	1	2	3	4	5	6	7	8	9	Total
Points:	15	8	8	23	10	15	10	5	10	104
Score:										

1. Match the following slope fields: (15)

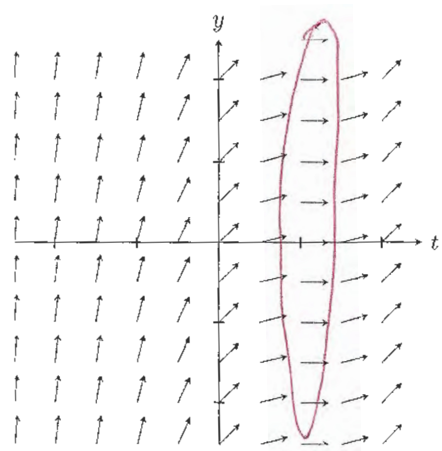
look for →



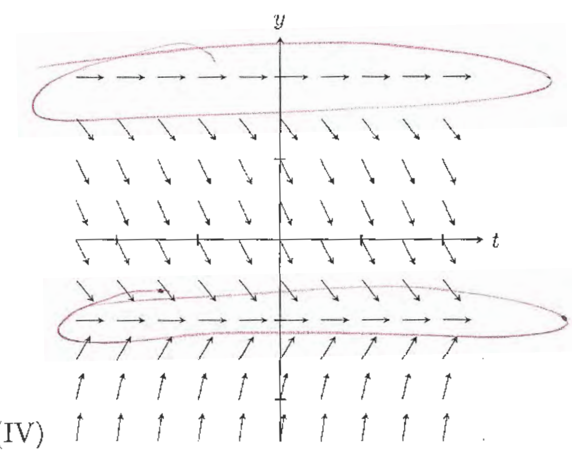
(I)



(II)



(III)



(IV)

(a) $\frac{dy}{dt} = (y-2)(y+1)$

IV ✓

(b) $\frac{dy}{dt} = (t-1)^2$

III

(c) $\frac{dy}{dt} = y(t+1)(y-1)$

II

(d) $\frac{dy}{dt} = (t-1)(t+5)$

None

(e) $\frac{dy}{dt} = -(y^2+1)$

I

Use $\boxed{v = \frac{y}{x}}$ $\boxed{\frac{dy}{dx} = x\frac{dv}{dx} + v}$

2. If $y = y(x)$ is a solution of (8)

$$y' = \frac{y}{x} + \frac{x}{y}, \qquad x > 0$$

then which **implicit** equation must it satisfy?

A. $y = x\left(\frac{x}{y} + C\right)$

B. $\ln y = \frac{x^2}{2} + x + C$

C. $y^2 = x^2 \ln x + C$

D. $y = x \ln x + Cx$

E. $y^2 = 2x^2 \ln x + Cx^2$

$\frac{dy}{dx} = \left(\frac{y}{x}\right) + \frac{1}{\left(\frac{y}{x}\right)}$

$x\frac{dv}{dx} + v = v + \frac{1}{v}$

$x\frac{dv}{dx} = \frac{1}{v}$

$\int v\,dv = \int \frac{1}{x}\,dx$

$\frac{v^2}{2} = \ln x + C$

$v^2 = 2\ln x + C$

plug $v = \frac{y}{x}$

$\frac{y^2}{x^2} = 2\ln x + C$

$\boxed{y^2 = 2x^2 \ln x + Cx^2}$

3. Find the implicit solution of the initial value problem: (8)

$$\underbrace{(6x^2y^2 + 4e^x - 2y\sin 2x)}_{M} + \underbrace{(4x^3y + \cos 2x)}_{N}\frac{dy}{dx} = 0, \qquad y(0) = 1.$$

A. $x^3y^2 + y\cos 2x + 4ye^x = 0$

B. $2x^3y^2 - y\cos 2x + 4e^x = 3$

C. $x^3y^2 + y^2\cos 2x + 4e^x = 5$

D. $\boxed{2x^3y^2 + y\cos 2x + 4e^x = 5}$

E. $2x^3y^2 + y\cos 2x - 4ye^x = -4$

check if Exact !

$M_y = 12x^2y + 0 - 2\sin 2x$

$N_x = 12x^2y - 2\sin 2x$ ✓

$\boxed{\psi_x = M} \Rightarrow \psi = \int (6x^2y^2 + 4e^x - 2y\sin 2x)\,dx + h(y).$

$\boxed{\psi = 2x^3y^2 + 4e^x + y\cos(2x) + h(y)}$

$\boxed{\psi_y = N} \Rightarrow \psi = \int (4x^3y + \cos(2x))\,dy + g(x) = \boxed{2x^3y^2 + y\cos(2x) + g(x)}$

$\therefore$ Gen Soln: $\boxed{2x^3y^2 + 4e^x + y\cos(2x) = C}$

Solve for C : $\boxed{y(0) = 1}$ $0 + 4 + 1 = C \Rightarrow \boxed{C = 5}$

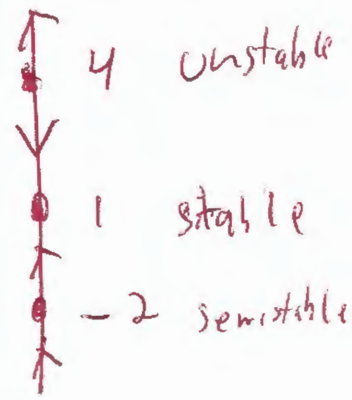
4. Consider the following differential equation $\frac{dy}{dt} = (y+2)^2(y-1)(y-4)$.

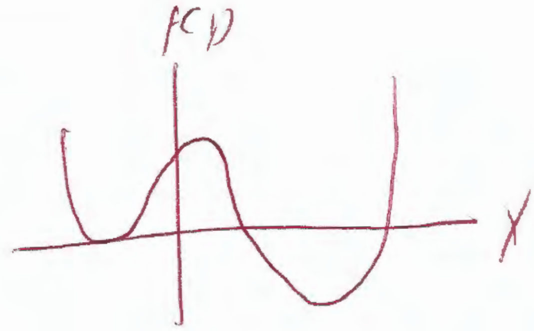
(a) Find the equilibrium solutions for this differential equation. (5)

(a)

$\boxed{y = -2, 1, 4}$

(b)

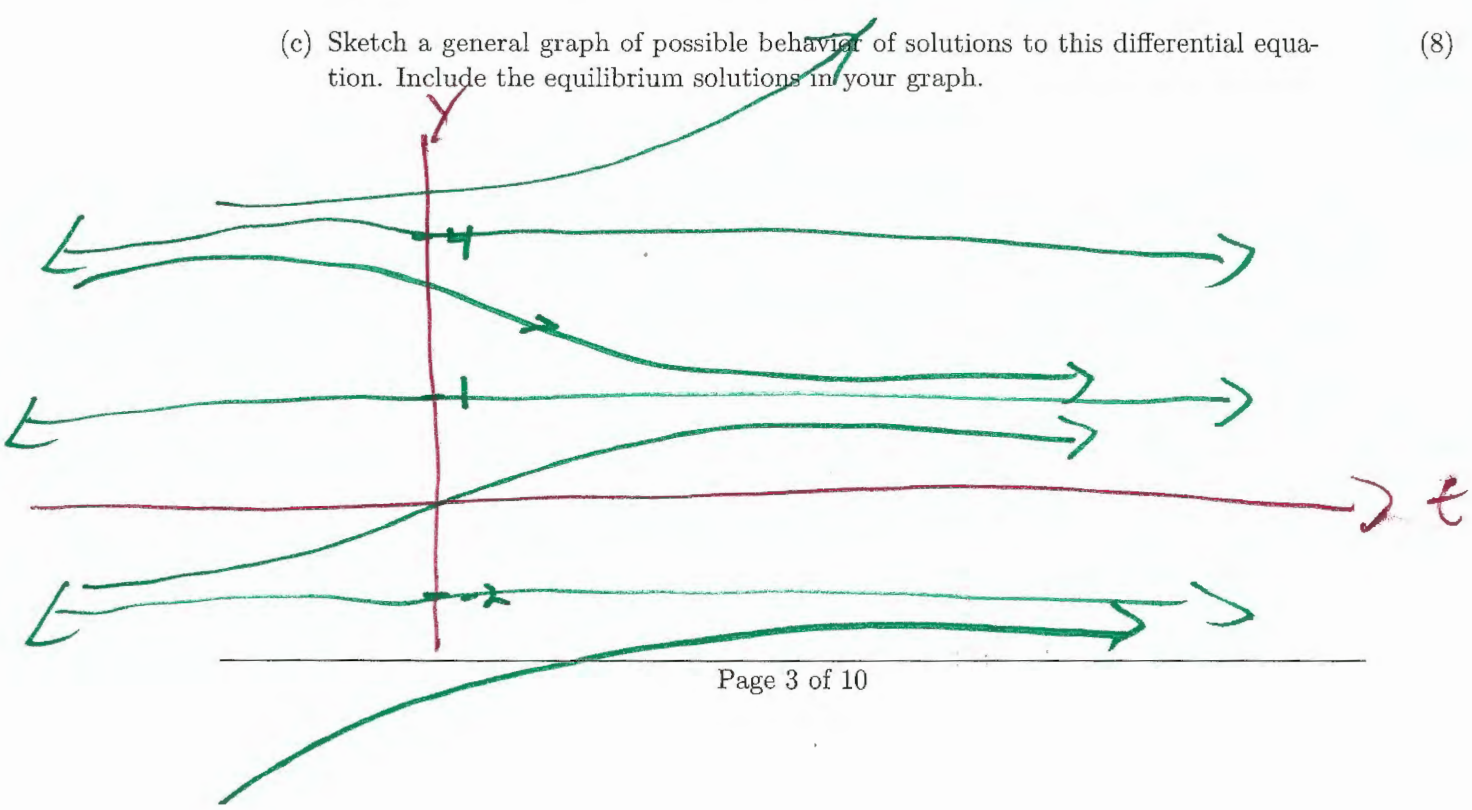




(b) Draw a Phase Line for this differential equation. Classify the equilibrium solutions. (10)

Oops

(c) Sketch a general graph of possible behavior of solutions to this differential equation. Include the equilibrium solutions in your graph. (8)



5. Consider the following initial value problem: (10)

$$\frac{dy}{dt} - y = \frac{\ln|20-4t|}{t^2-9}, \quad y(4) = -3.$$

For what interval can we can guarantee a unique solution *exists*?

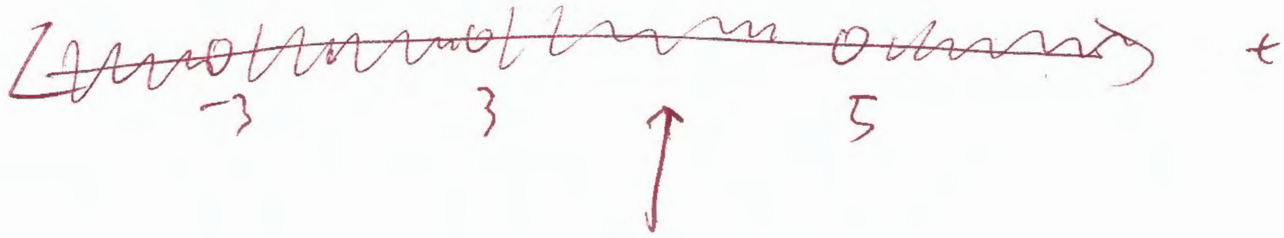
$p(t) = -1 \rightarrow$ always continuous

$g(t) = \dfrac{\ln|20-4t|}{t^2-9} \rightarrow$ continuous except on

bad points: $20 - 4t \neq 0 \Rightarrow t \neq 5$

$t^2 - 9 \neq 0 \Rightarrow t \neq \pm 3$

• p, g continuous on



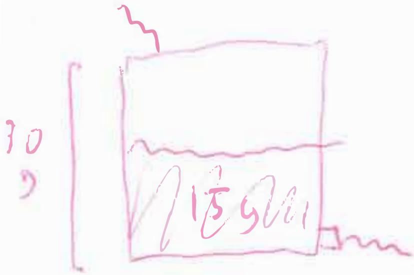
$t_0 = 4$

$\therefore$ $\boxed{I = (3,5)}$ is the biggest interval containing $t_0 = 4$ that guarantees unique soln to IVP above.

5. Suppose a 30-gallon tank contains 15-gallons of pure water. At time $t = 0$, we do two things: (15)

- we start pouring in 1 pound/gallon of sugar at a rate of 3 gallons/minute; and
- we open a drain at the bottom of the tank so that the mixture (sugar & water) pours out at a rate of 2 gallons/minute.

Write down an initial value problem modeling the amount $y(t)$ of sugar at time t, and solve it to find $y(t)$. How much sugar is in the tank when the tank is full?



$$\frac{dy}{dt} = \text{Rate in} - \text{Rate out}$$

$$= 1\,\frac{\text{lb}}{\text{gal}} \cdot 3\,\frac{\text{gal}}{\text{min}} - \frac{y}{15+t} \cdot 2\,\frac{\text{gal}}{\text{min}}$$

$y(t)$ = Amount of sugar at time t

$y(0) = 0$

Rearrange to get

$$\frac{dy}{dt} + \underbrace{\frac{2}{15+t}}_{g(t)}\, y = \underbrace{3}_{b(t)}$$

Integrating factor:

$$\mu(t) = e^{\int g(t)\,dt}$$

$$= e^{2\int \frac{dt}{15+t}}$$

$$= e^{2\ln|15+t|}$$

$$= e^{\ln(15+t)^2}$$

$$= (15+t)^2$$

$$y(t) = \frac{1}{\mu(t)}\left[\int g(t)\,b(t)\,dt + C\right]$$

$$y(t) = \frac{3\int(15+t)^2\,dt + C}{(15+t)^2}$$

$$y(t) = \frac{(15+t)^3 + C}{(15+t)^2}$$

$$y(t) = \frac{(15+t)^3 - 15^3}{(15+t)^2}$$

When $y(0) = 0$

$C = -15^3$

$y(15) = 26.25$ lb

7. Find the general solution to the following linear differential equations (10)

(a)

$$\frac{dy}{dt} + y = t^2 + \cos 5t.$$

$\boxed{y_h = ce^{-t}}$

Guess: $y_p(t) = at^2 + bt + c + d\cos 5t + e\sin 5t$

LHS:

$\frac{dy_p}{dt} + y_p = \frac{d}{dt}\left[at^2 + bt + c + d\cos 5t + e\sin 5t\right]$

$+ at^2 + bt + c + d\cos 5t + e\sin 5t$

$[2at + b - 5d\sin 5t + 5e\cos(5t)]$

$+ at^2 + bt + c + d\cos 5t + e\sin 5t$

$= (a)t^2 + (2a+b)t + (b+c)$

$+ (5e + d)\cos 5t + (e - 5d)\sin 5t$

$\overset{?}{=} 1t^2 + 0\cdot t + 0 + 1\cdot\cos 5t + 0\cdot\sin 5t$

Match coefficients

$a = 1$ $\quad\boxed{a=1}$

$2a + b = 0$ $\quad\boxed{b=-2}$

$b + c = 0$ $\quad\boxed{c=2}$

$5e + d = 1$

$e - 5d = 0$

$\boxed{d = \frac{1}{26}}$

$\boxed{e = \frac{5}{26}}$

General solution

$y(t) = ce^{-t}$

$+ t^2 - 2t + 2$

$+ \frac{1}{26}\cos 5t + \frac{5}{26}\sin 5t$

(b)

$$\frac{dy}{dt} + \underbrace{2t}_{g(t)}y = \underbrace{4e^{-t^2}}_{\to h(t)}.$$

Integrating Factor

$$\mu(t) = e^{\int g(t)\,dt}$$
$$= e^{\int 2t\,dt}$$
$$= e^{t^2}$$

Multiply $\mu(t)$ out

$$e^{t^2}\frac{dy}{dt} + 2te^{t^2}y = 4\cancel{e^{-t^2}} \cdot \cancel{e^{t^2}}$$

$$\frac{d}{dt}\left[e^{t^2}y\right] = 4$$

$$\int \frac{d}{dt}\left[e^{t^2}y\right]dt = \int 4\,dt$$

$$e^{t^2}y = 4t + C$$

$$y = e^{-t^2}\left[4t + C\right]$$

(c)

$$\frac{dy}{dt} + \frac{2}{t}y = t - 1 + \frac{1}{t}.$$

$M(t) = e^{\int \frac{2}{t} dt}$

$= e^{2\ln|t|}$

$= \cancel{e}^{\ln t^2}$

$= t^2$

Multiply $M(t)$ out

$t^2 \frac{dy}{dt} + \frac{2}{t} t^2 y = t^2\left(t - 1 + \frac{1}{t}\right)$

$\frac{d}{dt}[t^2 y] = t^3 - t^2 + t$

$\int \frac{d}{dt}[t^2 y] dt = \int (t^3 - t^2 + t) dt$

$t^2 y = \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} + C$

$$\boxed{y = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{c}{t^2}}$$

8. Find the general solution to the following separable differential equation (5)

(a)

$$\frac{dy}{dt} = 6y^2t.$$

$$\int y^{-2}\,dy = \int 6t\,dt$$

$$\frac{y^{-1}}{-1} = 3t^2 + C$$

$$-\frac{1}{y} = 3t^2 + C$$

explicit

$$\boxed{y(t) = \frac{-1}{3t^2 + C}}$$

(b)

$$\frac{dy}{dt} = \frac{t}{y\,(1+t^2)^3}$$

$$u = 1+t^2$$

$$du = 2t\,dt$$

$$\frac{du}{2} = t\,dt$$

$$\int y\,dy = \int \frac{t}{(1+t^2)^3}\,dt$$

$$\frac{y^2}{2} = \frac{1}{2}\int u^{-3}\,du$$

$$\frac{y^2}{2} = \frac{1}{2}\,\frac{u^{-2}}{-2} + C$$

$$\frac{y^2}{2} = \frac{-1}{4}\cdot\frac{1}{u^2} + C$$

$$\frac{y^2}{2} = -\frac{1}{4}\cdot\frac{1}{(1+t^2)^2} + C$$

$$y^2 = -\frac{1}{2}\cdot\frac{1}{(1+t^2)^2} + C$$

$$\boxed{y = \pm\sqrt{-\frac{1}{2}\cdot\frac{1}{(1+t^2)^2} + C}}$$

9. Consider the following initial value problem, (10)

$$\frac{dy}{dt} = y^2 + t^3, \quad y(0) = 1.$$

By hand, use Euler's method with $h = 1$ (this is a horrible h) to approximate the value of y_2. Use the table to record the necessary values of k,t, y, and $dy/dt = f(t, y)$ at each step. Show all work if you want to receive credit.

k	t_k	y_k	$f(t_k, y_k)$
0	0	1	$1^2 + 0^2 = \boxed{1}$
1	1	$y_1 = y_0 + f(t_0, y_0)h$ $= 1 + 1 \cdot 1$ $= \boxed{2}$	$2^2 + 1^3$ $= \boxed{5}$
2	2	$y_2 = y_1 + f(t_1, y_1)h$ $= 2 + 5 \cdot 1 = \boxed{7}$	